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Report on the I.U.T.A.M. symposium on rotating fluid systems

By F. P. BRETHERTON
Department of Applied Mathematics and Theoretical Physics, Cambridge University

G. F. CARRIER
Pierce Hall, Harvard University

AND M. S. LONGUET-HIGGINS
National Institute of Oceanography, Wormley, Surrey

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A symposium on 'Rotating fluid systems' was held at La Jolla, California, from 28 March to 1 April 1966. The meeting was organized under the auspices of the International Union of Theoretical and Applied Mechanics, and took place at the Institute of Geophysics and Planetary Physics of the University of California. There were 75 participants, and attendance was by invitation. The subjects discussed included steady flows, both thermally driven and non-thermal; transient motions and instabilities; planetary waves and gravity waves affected by rotation; inertial oscillations; and hydromagnetic flows relating to the Earth's interior. The following is a brief account of the proceedings as seen by the authors. No formal volume of the papers presented is to be published, but references to published or unpublished work are given at the end of this article.

1. Introduction

Fluid systems under rotation have been studied in principle since the days of Newton and Laplace. But only since the theoretical work of Taylor and Proudman in this century, supported by Taylor's ingenious experiments, have the peculiar effects of rotation on a fluid system been generally appreciated. Earlier studies on the stability of gravitating masses were inspired partly by applications to cosmical dynamics. More recently, however, the stimulus for research has come mainly from geophysical applications: to the atmosphere and oceans, and to motions within the Earth's core. Some applications are also to be found in fluid engineering.

The General Assembly of the International Union of Theoretical and Applied Mechanics at Munich in September 1964 approved a suggestion made originally by Prof. G. K. Batchelor that a Symposium be held during 1966 on the subject of 'rotating fluid systems'. The committee which was subsequently appointed by IUTAM to organize the symposium was as follows: G. F. Carrier (Harvard) Chairman; A. Eliassen (Oslo); H. Görtler (Freiburg); M. A. Lavrentiev (Novosibirsk); M. S. Longuet-Higgins (Wormley) Secretary; J. W. Miles (San Diego); B. Saint-Guily (Paris). At the invitation of Professor W. H. Munk, the Symposium was held at the Institute of Geophysics and Planetary Physics at La Jolla, in its elegant new building overlooking the Pacific Ocean.

The subjects for discussion were defined by the Committee as follows:

1. Steady flows in rotating fluids, including boundary flows and jets.
2. Instabilities in (1).
3. Transient motions and adjustment to steady states.
4. Inertial oscillations and their stability.
5. Planetary waves (Rossby waves), and gravity waves affected by rotation.
6. Hydromagnetic phenomena in rotating fluids.

This classification was generally followed in the organisation of the meeting. Several other topics of interest, including the stability of gravitating masses, could unfortunately not be included for lack of time. The conference was concerned with fields of uniform or nearly uniform rotation, concentrated vortices having received special attention at the IUTAM meeting at Ann Arbor in 1964 (see Küchemann 1965). Participation in the meeting was by invitation, and was limited to those taking an active interest in the subject. The total number, 75, was determined by the facilities available at the I.G.P.P.

Besides the research papers, of which there were 45 (compared with 70 at the Ann Arbor meeting), each session contained at least one introductory paper. In particular during the first session M. J. Lighthill gave an introductory survey of the whole field under discussion. His paper is printed in full in this number of *Journal of Fluid Mechanics*.* The remaining papers were organized in six sessions (covering 5 days) roughly corresponding to the headings given above, but with certain exceptions. In describing the papers we shall follow the order in which they were presented, with some exceptions.

We should like here to acknowledge the benefit we have gained from discussions with other participants at the Symposium, and from access to some remarkably complete and accurate notes made by Prof. D. Fultz. For any views expressed in this report we take of course the sole responsibility.

2. Opening session

In the opening session of the symposium Prof. W. H. Munk expressed a cordial welcome to the participants, and explained briefly the role of the Institute of Geophysics and Planetary Physics in the Scripps Institute of Oceanography and in the University of California. He reminded them of the experimental and theoretical work in progress at the Institute on deep sea tides, and described briefly preliminary measurements from a new instrument which records with great sensitivity pressure and temperature at a height of 2 ft. above the ocean bottom in water of depths of 7000 ft. Besides the tidal variations of pressure, there are also intermittent bursts of rapid fluctuations of temperature. These bursts are apparently roughly related to the tidal cycle, but their interpretation awaits

* An asterisk will denote a reference to the papers presented at the symposium which are listed at the end of the report.

further investigation of the bottom boundary layer of the ocean, which is hitherto unexplored.

M. J. Lighthill* then gave a wide-ranging survey of present knowledge of the dynamics of rotating fluids, using vorticity as the thread connecting different aspects, and emphasizing the importance of the basic degeneracy in geostrophic motion. The statement that the Coriolis forces on a fluid element are balanced by the pressure forces does not provide a closed system of equations from which it is possible to predict future development of a flow, and for flows at low Rossby number it is necessary to consider higher-order effects to determine the motion. These may be inertial, viscous, or geometrical in character, and it is necessary in each problem to assess their relative importance.

3. Steady flows

The first research paper was given by R. Hide *a** describing flows due to sources and sinks in a homogeneous liquid in a rotating container bounded by rigid plane end-plates perpendicular to the axis of rotation. Experiments at low Rossby number with several different geometries showed that there are two distinct classes of possible velocity field. In the first the motion is almost everywhere two-dimensional and effectively independent of the rotation of the system. That these two effects are characteristic of flow at low Rossby number was shown by Taylor (1917, 1921) and Proudman (1916). The volume transport from source to sink takes place throughout the interior of the fluid, and the velocities are small. In the second class, the volume transport is almost all in the Ekman boundary layers on the bounding end-plates, whereas in the interior a strong gyre appears which has no vorticity relative to the rotating container. In discussion L. N. Howard pointed out that the criterion which was advanced by the speaker for the occurrence of the different classes of motion could be understood more simply by considering the conditions under which the Coriolis term, as well as the centrifugal term in the equation of motion, can be written as the gradient of a scalar variable and absorbed into an effective pressure. This may only be done if the stream function describing the two-dimensional interior motion is single-valued. This is always so in a simply connected region, but in a multiply connected region only if the sources and sinks are in pairs in close proximity, i.e. as dipoles. Then a two-dimensional motion unaffected by rotation is possible and Taylor's theorem holds. With this geometry the observed motion was always of the first class. If, on the other hand, a closed ring of fluid in the interior region surrounds a source, no pressure gradients can counteract its tendency to spin relative to the rotating system as it expands. This relative circulation causes a large outward volume flux in Ekman layers on the end-plates, and the relative vorticity in the interior is removed by Ekman-layer suction. In this case the observed velocity field was always of the second class. A. J. Faller suggested that, if the end-plates had not been quite parallel, quite different results would have been observed.

T. Maxworthy* then described measurements of the velocities in a homogeneous liquid contained between a stationary horizontal circular plate and a parallel disc above rotating in its own plane and acting as a centrifugal pump. The exact solution of the Navier–Stokes equations obtained by Bödewadt (1940) for a disk of infinite radius predicts a vertical velocity which is independent of distance from the axis of rotation, and the azimuthal and radial velocities have a definite structure. Although measurements of the radial and azimuthal velocities for a disk of finite radius were in fairly good agreement with the theory, on closer investigation it appeared that the boundary layer on the lower stationary plate was separating as a jet along the axis of rotation, which a short distance above the plate underwent a change similar to vortex breakdown (see figure 1, plate 1). Even when the disc radius was much larger than the separation between the disc and the plate, a similar non-uniformity in the vertical velocity was observed.

The last paper in the opening group on steady motions was presented by C. S. Yih*. He described an extension of the Taylor–Proudman theorem to weak steady motion of a stratified fluid in rotation about a vertical axis. When the Rossby number (the ratio of the inertial to the Coriolis forces) is small, and the fluid density ρ is uniform over surfaces of constant potential for the combined gravitational and centrifugal force fields but varies with height z, the condition for geostrophic motion requires that

$$(2\mathbf{\Omega}.\nabla)\rho\mathbf{u} = \mathbf{0}.$$

He also predicted that under certain circumstances the diameter of the Taylor column above a body would, when the fluid is stratified, increase with height. In discussion, some doubts were expressed about the relevance of this solution when slight viscosity or time dependence were present.

The next group of papers were on the steady circulation in the oceans. V. M. Kamenkovich* discussed a model on a beta-plane in a square ocean basin, in which the stream function Ψ satisfies the equation

$$\epsilon(\Psi_x\nabla^2\Psi_y - \Psi_y\nabla^2\Psi_x) - \gamma\nabla^4\Psi + \Psi_x + g(y) = 0.$$

Here $g(y)$ describes the curl of the wind stress on the surface of the ocean and had the form $4y(1-y)$, and Ψ_x accounts for the rate of variation of the Coriolis parameter following a fluid particle. The non-linear terms describe the advection by the fluid of relative vorticity $\nabla^2\Psi$, and $\gamma\nabla^4\Psi$ the mixing of vorticity by horizontal eddy diffusion. The coefficients ϵ and γ are small. Dr Kamenkovich presented numerical and analytical calculations of the boundary-layer equation, which showed that, for large Reynolds number ($R > 10$), where

$$R = \epsilon^{\frac{1}{2}}/\gamma,$$

the solutions for a mixed inertial–viscous boundary layer on the south-west coast boundary ($y < 0$) cannot be continued into the northern part ($y > 0$). A singularity first occurs on $y = 0$ in the interior of the boundary layer. A similar upper limit is found for R if there are solutions of a boundary-layer type in the north-west corner.

A. R. Robinson* described an extensive analysis, performed in collaboration with P. P. Niiler* and S. L. Spiegel*, of a stratified inertial current, either on the western boundary of the ocean or as a free jet. Taking curvilinear co-ordinates following the core of the current, and making a scale analysis in terms both of the length scale of meanders of the core, and of major changes in the structure, he presented velocity and density profiles which depend on the assumption of the vertical distribution of density and potential vorticity at the outer edge of the current. These seem realistic if the potential vorticity is taken to be uniform. These solutions demand that the inflow into the free jet from the interior of the ocean is linked to the meanders, although the detailed coupling was not discussed. Following calculations similar to those of Warren (1963), but based on new data about the ocean depth, he claimed that the hypothesis that the path of the Gulf stream is controlled by bottom topography after it leaves the continental shelf at Cape Hatteras is not consistent with reasonable postulates about the structure and mass transport in the stream.

D. J. Baker and A. R. Robinson* described an elegant model in the laboratory, in which a liquid is contained in a portion of a spherical shell, which is mounted in various altitudes on a rotating turntable. Motion in the liquid is driven by rotating one of the spherical bounding surfaces about a radius. The dimensions of the apparatus were chosen so as to model as far as possible the motions of a homogeneous ocean in a closed circular basin on a beta-plane. The forced motion was made visible using changes in acidity in the liquid in conjunction with a pH indicator. A strong intensification of the velocities was observed on the equivalent western boundary in the model. Whenever the model was mounted so that the equivalent ocean basin intersected the equator, an eastward motion was observed there. However, when the sense of the driving motion was reversed, the velocity pattern reversed too, at least when the amplitude was not too large, suggesting that the model dynamics were linear.

After tea on the first day, K. Hidaka* presented a computation of planetary flow in the equatorial current system of the Pacific, and D. W. Moore & P. G. Drazin* described a modified version of Taylor's classical experiment on flow past a body in a rotating cylinder of water. G. K. Batchelor presented a paper on behalf of G. I. Taylor* entitled 'Motion of Solid Bodies in Rotating Fluids'. In this he utilized a theorem due to Taylor (1917), that for an incompressible liquid in two-dimensional motion the entire effect of a superposed solid-body rotation may be described by an equivalent pressure, to discuss the motion of a cylinder of density ρ' different from that of the liquid. If $\rho' > \rho$ the cylinder spirals outward from an equilibrium position. If $\rho' < \rho$ it spirals inwards.

4. Thermally driven flows

The second day of the conference was devoted to thermal convection in a rotating fluid. In his introductory talk, Hide discussed the Bénard problem in a system rotating about a vertical axis and remarked on the qualitative effects of rotation on the stability to infinitesimal disturbances of the state of no relative motion. The Taylor–Proudman theorem explains qualitatively the increase in height-to-width ratio for the most unstable cell, and why the Rayleigh number

$$R_a = g\alpha\Delta T \,.\, d/\kappa\nu$$

is larger. Here g is the acceleration due to gravity, α, κ and ν are the expansion coefficient, thermal diffusivity and kinematic viscosity of the fluid, d and ΔT are the depth and temperature excesses between the upper and lower boundaries. For small values of the Taylor number

$$T_a = 4\Omega^2 d^4/\nu^2$$

the imposition of a rigid boundary stabilizes the system relative to a free surface there, but at large Taylor numbers the greater horizontal convergence in the Ekman layer at a rigid surface may result in greater instability. Unstable oscillations may also arise.

In a later session G. Veronis* described a simplified analysis of a fluid bounded above and below by a free surface, in terms of truncated Fourier series. He predicted that, for a range of Taylor numbers up to about 10^4 or larger, steady two-dimensional disturbances of finite amplitude should exist, at Rayleigh numbers smaller by a factor of 2 than the appropriate critical number indicated by infinitesimal theory. He gave a qualitative argument showing why this was so.

H. T. Rossby* then described a series of very careful experiments on Bénard convection, first of all in a non-rotating system (see figure 2, plate 2). Care was taken to ensure that the temperature difference between the two boundaries was independent of time, and restricted in magnitude to less than 5 degC. Measurements of the Nusselt number N (the ratio of the actual heat transfer in a given vessel to the transfer on the assumption that it is entirely conductive) showed that for large Rayleigh numbers

$$N \propto R^{0\cdot 28}$$

for water, but

$$N \propto R^{0\cdot 25}$$

for mercury. He also presented graphs of the critical Rayleigh number and the Nusselt number versus the Taylor number. There are apparently circumstances in which the effect of increasing the rotation is to increase the transfer of heat. A finite-amplitude subcritical instability for Taylor number less than 10^4 was confirmed.

V. Barcilon* considered the weak steady motion forced by a heat source localized in a horizontal plane boundary beneath a semi-infinite stably stratified fluid. The kinematic viscosity ν and heat diffusivity κ were assumed uniform, and the motion was assumed to satisfy the Boussinesq approximation. He showed that the linearized equations form an eighth-order system, which decouples into a sixth-order system and Laplace's equation. There are two distinct scales on which the disturbance penetrates into the liquid; the scale L of the imposed variations at the boundary, and the Lineykin length $(\kappa/\nu)^{\frac{1}{2}}\, 2\Omega L/N$, where N is the Brunt–Vaisälä frequency of the basic state (assumed uniform) and Ω the rate of rotation about a vertical axis. Each scale is associated with a different combination of velocity and perturbation temperature gradients. There is also an Ekman layer on the rigid boundary. If, however, one allows diffusion of heat and momentum in a vertical direction only, the penetration to depth L does not appear at all, so care is necessary in developing theories of the oceanic thermocline along the lines initiated by Lineykin (1955).

Another group of papers were concerned with the flow in a rotating annulus of which the outer wall is heated and the inner wall cooled. In his introductory talk R. Hide had referred to a survey article by Fultz (1961) and a review of the theory by Brindley (1964). He then explained the significance of the dominant parameters. These are the thermal Rossby number

$$R_0 = g\alpha\Delta T . d/4\Omega^2(b-a)^2.$$

(where a, b are the radii of the inner and outer walls of the annulus and ΔT the temperature difference between them), the Ekman number

$$E = \nu/2\Omega(b-a)^2$$

and the Prandtl number $P = \nu/\kappa.$

Of possible importance, also, is the aspect ratio $(b-a)/d$. An internal parameter useful in interpreting the motion is

$$B = -g\alpha(\partial T/\partial z)\, d^2/4\Omega^2h^2,$$

where $\partial T/\partial z$ is a vertical temperature gradient which is set up as a result of the motion. He stated that a simple physical argument showed that $\partial T/\partial z = \frac{2}{3}\partial T/\partial x$ in the interior of the fluid, and showed some numerical calculations by S. A. Piacsek* confirming this. From measurements in a new apparatus much larger than those previously used, it appears that the limiting value of R_0 as E^{-2} tends to infinity for which the transition occurs from the upper symmetric régime to the wave régime may not after all be inconsistent with assuming that the transition is described by the Eady model of baroclinic instability, but no existing theory correctly predicts the remainder of the transition curve.

D. Fultz* then described work in progress at Chicago, making a critical evaluation of present experimental data on this system. The principal conclusions were that all existing data for transition between the symmetric and the wave régime could be fitted on a single curve connecting

$$R_0[d/(b-a)]^{\frac{1}{2}} \quad \text{and} \quad E^{-2}[d/(b-a)]^{-\frac{1}{2}}.$$

However, H. A. Snyder* in a later session claimed that, following experiments in an annulus of very small aspect ratio, a single formula of this sort was untenable. Fultz also remarked that, if sufficient care is taken, there is no hysteresis on the lower symmetric curve, and a fine structure has emerged in the transitions in the wave régime.

J. Brindley* put forward some qualitative remarks about the flow in the upper symmetric régime, based on the hypothesis that in the interior of the fluid the governing equations are the thermal wind relation, and that temperature and angular momentum about the axis are constant following a fluid particle. Information on this point is provided by numerical computations of the complete equations for the axisymmetric flow in five cases by S. A. Piacsek. In discussion F. P. Bretherton stated that preliminary results from an almost identical numerical programme by C. Quon at the University of Cambridge showed that for Prandtl number 1 (as opposed to 7) substantial conduction of heat takes place in the interior, although the motion is qualitatively similar. R. H. Rogers described some theoretical work in progress at Brighton on the wave régime of the rotating annulus problem.

5. Transient motions

In reviewing the early studies of transient motions in rotating inviscid fluids K. Stewartson emphasized the lack of uniqueness which arises in such studies, and noted that the resolution of such difficulties via the treatment of initial value problems does not necessarily lead to the same predictions as those found for the small viscosity limit of the flow of a viscous fluid. He discussed, in particular, the formation of a Taylor column by an axially moving object and the flow past the Taylor column implied by the presence of a topographical bump in the container. In describing the motion associated with spinning disks he noted particularly the observed asymmetric flow which ultimately arises when two coaxial disks are given angular velocities Ω and $-\Omega$.

In the course of discussion F. P. Bretherton remarked that a recent reinterpretation of the initial value problem for a slowly moving obstacle showed inertial waves of all frequencies up to 2Ω propagating outwards in the appropriate directions and with the appropriate group velocities. The developing Taylor column could be ascribed to inertial waves of zero frequency (but non-zero velocity) propagating parallel to the axis of rotation. The first waves to arrive at a point are those of large length-scale, the development of the singularities at the edge of the column being due to the later arrival of waves of diminishing scale. The theory can easily be extended to take account of viscous effects.

L. M. Hocking* analysed the flow which would occur when a cylindrical shell of fluid whose initial angular speed is $\Omega(r)$ was subjected to an additional external pressure. He noted that in many cases the resulting motion would be oscillating and that collapse of the central cavity would not always ensue.

J. Pedlosky* recalled that a mass of fluid of depth L will 'spin up' after a sudden change in container speed in a time of order $L/(\Omega\nu)^{\frac{1}{2}}$. By a detailed analysis, he showed that two immiscible fluid layers (of comparable viscosity) in that container would spin up more slowly but in a time which again is of order $L/(\Omega\nu)^{\frac{1}{2}}$. He stressed the important role of the interface Ekman layer. He then discussed qualitatively the spin up of a continuously stratified medium suggesting that there was an essential difference between this and the multilayer case. A new side-wall boundary layer, absent in both the multi-layer model and in the homogeneous case, is present instead of the more familiar layers parallel to the axis rotation found by Stewartson, Greenspan and Howard. This layer is not taken into account in a recent analysis by Holton (1965). Without side-wall boundary layers the horizontal average of the relative vorticity outside the bottom Ekman layer would never change.

H. J. Lugt & E. W. Schwiderski* applied Stokes's equations of slow motion to study numerically the diffusion of vorticity in a rotating cylindrical container. The ends of the container being ignored, the same solution applies whether or not the fluid is rotating.

J. Charney reviewed the early history of studies in the stability of rotating fluids. The interface between two homogeneous liquids of slightly different densities and velocities in a rotating system is inclined at an angle to the horizontal, and provides a model of a front in the atmosphere. He remarked on our incomplete understanding of the stability of this situation. No theory exists for some experiments by Fultz in which an isolated mass of liquid resting on a rigid boundary is surrounded by a less dense liquid which is rotating faster. Kotschin (1932) had considered the case in which the front is plane, and the system is bounded above and below by rigid horizontal surfaces, but, according to recent work by Stone at Harvard, there is an error in his analysis.

Charney then defined the 'baroclinic stability problem' for quasi-geostrophic motions with continuous variations of density. He explained that when the basic velocity depends on height only it may be reduced to an eigenvalue problem similar to that in Rayleigh's analysis of the stability of inviscid uni-directional flow in a non-rotating homogeneous liquid. The boundary conditions are, however, different, and the detailed mathematics is complicated. Incorrect conclusions were at first drawn from several partial analyses, but a more complete treatment has been given in a sequence of papers by Miles (1964*a*–*d*, 1965). Making use of an integral theorem, Charney then explained qualitatively how density gradients at the horizontal bounding surface may be associated with instability. He also remarked that such density gradients at the ground are necessary within non-linear quasi-geostrophic theory if there is to be a cascade of energy into motions of smaller and smaller length scale, and suggested this might be relevant to the formation of sharp fronts.

G. Arnason gave the results of an analysis of the Eady baroclinic stability model for small Richardson number.

Two papers followed which dealt with the instability of Ekman boundary layers. Using optical methods of observation, A. J. Faller & R. E. Kaylor* identified two distinct modes of boundary-layer instability, each consisting of two-dimensional roll vortices superposed upon the basic flow. The first type have their axes oriented at about 14° to the left of the direction of basic flow, are stationary and have a relatively short wavelength. The second type are oriented at angles varying between 20° to the right and 5° to the left of the basic flow, the vortices move rapidly (with speeds dependent on their orientation) and their spacing is 2 to 3 times that of the first type. The second type of instability occurred at somewhat lower Reynolds number, but the rate of growth was sometimes exceedingly slow. Photographs illustrated each of the types of waves in their different régimes and their interaction when they occurred simultaneously (see figure 3, plate 3). The theory of the type 1 disturbances was well understood; they occurred mainly within the Ekman layer itself. The type 2 instabilities seemed to extend further into the main flow. Numerical calculations were not in disagreement with the observed results. The transition between type 2 instabilities and turbulent flow had been studied experimentally.

P. R. Tatro* described some observations of Ekman-layer instabilities made with hot-wire anemometers. The results for critical Reynolds number were in very good agreement with those of Faller and Kaylor. He pointed out that the fluctuations associated with type 2 instability can extend far into the geostrophic flow, and preseumably can be regarded as inertial waves thrown off by the fluctuations in the boundary layer.

H. A. Snyder* described the results of an extensive set of experiments in a cylindrical annulus in which the rates of rotation Ω_1 and Ω_2 of the inner and outer cylinder, and the corresponding temperatures θ_1 and θ_2 were individually varied. The gap was wide ($b = 2a$). In the case when θ_1 and θ_2 were equal and $\Omega_1 = 0$, and in the range of Taylor numbers lying between the critical value at which Taylor vortices were first observed and 10^3 times this value, no azimuthal dependence was observed and no transition to turbulence occurred. The experimental results observed in the case $\theta_1 \neq \theta_2$ have been mentioned earlier (p. 399).

W. H. Reid* gave a preliminary characterization of the stability problem associated with the configuration in which a homogeneous fluid between two axial rotating cylinders has an axial flow at Reynolds number of order 10^3. He pointed out that asymptotic methods developed for parallel shear flows are generally applicable but with certain modifications.

6. Planetary waves, and gravity waves affected by rotation

In an introductory paper F. P. Bretherton described the basic mechanism for planetary wave motions, namely the variation in vorticity of the fluid due to the stretching of vortex lines, and how this accounted for the westward phase velocity. He described some of the known wave solutions on a sphere or β-plane covered with fluid to uniform depth, including solutions for closed basins. He indicated some of the effects of stratification, and estimated the importance at the equator of omitting Coriolis forces arising from the horizontal part of the earth's rotation vector.

He emphasized, however, that such solutions may be very sensitive to changes in depth, or indeed to any other disturbing influence (this being a particular example of the degeneracy mentioned earlier by Lighthill*). Further, the baroclinic Rossby waves in the ocean have such small phase velocities that they must be strongly affected or distorted by quite weak ocean currents.

The first two research papers in this session dealt with the theoretical problem of how the normal mode oscillations on a sphere behave at high rates of rotation, that is to say for large values of the parameter $\epsilon = 4\Omega^2R^2/gh$. In a paper read by G. S. Golitsyn* he and L. A. Diky had found by direct numerical calculation that when $\epsilon \gg 1$ the waves travelling towards the west fell into two distinct classes, having frequencies asymptotically proportional to $\epsilon^{-\frac{1}{4}}$ and $\epsilon^{-\frac{1}{2}}$ respectively. By the use of a WKB approximation it was possible to derive the leading terms in the corresponding asymptotic expansions. However, the two types of wave did not completely correspond to the waves of the first and second class at low rates of rotation. For each value of the longitudinal wave-number s, one mode of the first type when $\epsilon \gg 1$ went over to a mode of the second class when $\epsilon \ll 1$. At high rates of rotation the energy tended to become concentrated towards the equator.

Similar though more complete results were reported by M. S. Longuet-Higgins,* who found that the eastwards-travelling modes all had frequencies proportional to $\epsilon^{-\frac{1}{4}}$ at high rates of rotation, but with one significant exception; for each value of s, one of the eastward modes went over into a Kelvin wave trapped at the equator, and with frequency proportional to $\epsilon^{-\frac{1}{2}}$ (see figure 4). An analytical theory could be given and it was pointed out that to a first approximation the waves could be described by an equatorial β-plane approximation. However, one of the solutions predicted by the β-plane approximation was not realized on the sphere.

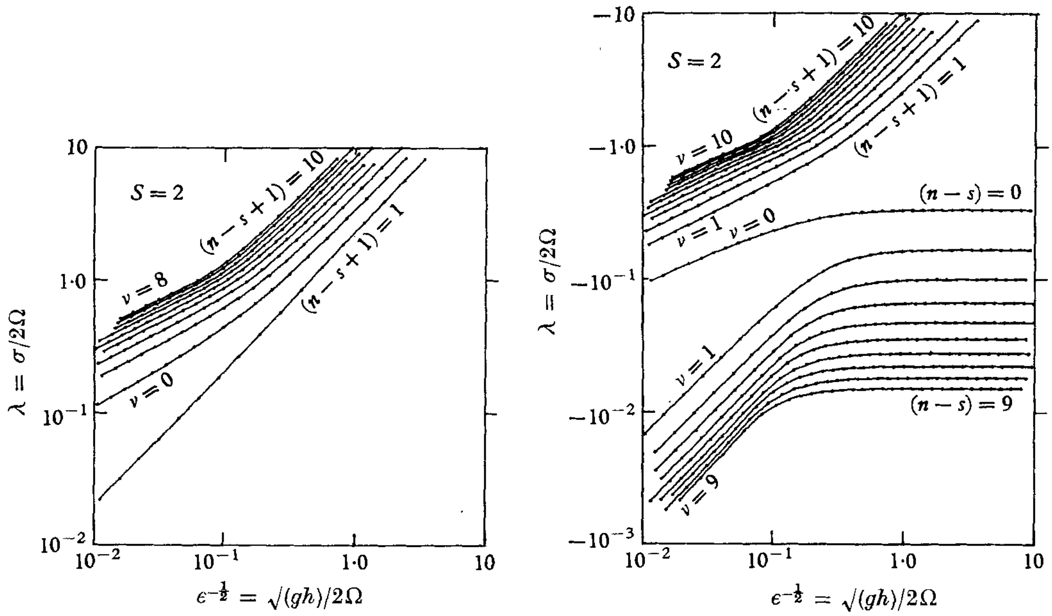

FIGURE 4 (M.S. Longuet-Higgins). Frequencies of the free modes of oscillation of fluid in a thin layer covering a rotating globe of unit radius, as a function of $\sqrt{(gh)}/2\Omega$. (*a*) Modes travelling eastwards, (*b*) modes travelling westwards. The eastwards mode of lowest frequency corresponds to a Kelvin wave. The longitudinal wave-number s is equal to 2.

These two papers seem to emphasize not only the heuristic value of computation but also the advantage to be gained from greater accuracy; for the asymptotic behaviour of the eastwards modes only became apparent when the expansion of the eigenfunctions in spherical harmonics was carried to 40 terms or more.

The next two papers were concerned with the detection or the prediction of planetary waves within the atmosphere and ocean, respectively.

The existence in the atmosphere of planetary waves stationary with respect to the Earth's surface has been known since the time of Rossby. However, B. Machenhauer* (see Eliassen & Machenhauer 1965) described an interesting investigation directed towards analysing the non-stationary part of the large-scale pressure field. For the winter of 1956–57, the changes over 24 h of the 500 mb and 1000 mb surfaces over the northern hemisphere were analysed into spherical harmonics. The coefficients of the lower harmonics had the characteristic form of travelling planetary waves, and the speeds of propagation of the lower harmonics agreed well with the theoretical speeds of propagation, especially when the effect of horizontal convergence was taken into account.

M. Rattray* described a theoretical prediction of the periods of free planetary-wave modes within the major ocean basins of the world. The Atlantic, Pacific and Indian oceans were each idealized as a rectangular basin of uniform depth on a β-plane, f being allowed to vary linearly with latitude. The eigenfunction for the unbounded plane can be expressed in terms of Weber functions; and with certain assumptions as to the geostrophic nature of the flow these may be fitted so as to satisfy the boundary conditions. The effect of divergence was included. There seemed little reason why the calculated periods of the *lower* modes should be changed by detailed variations in topography or coastline. As pointed out in the discussion, the periods of the lowest free modes for the Pacific Ocean agree well with those for a circular basin centred on the equator (Longuet-Higgins 1966). In the latter paper some observational evidence is given for the existence of the lowest mode in the Pacific Ocean. H. P. Greenspan commented that he had checked experimentally the frequencies of planetary-wave oscillations in a rotating cylindrical vessel with a sloping bottom.

In view of the probable influence of non-linear effects and of uneven bottom topography on Rossby waves the next two papers were of special interest. N. P. Fofonoff* described a new class of non-linear solutions to the Rossby-wave equations. These were obtained by an initial transformation to Lagrangian co-ordinates, taking the stream function ψ and the time t as independent variables. Terms representing wind stress and friction were included. In the case of a constant curl of wind stress solutions can be obtained of the form

$$x = a_0 + a_1\psi, \quad y = b_0 + b_1\psi,$$

where the coefficients a_i and b_i are functions of t. (Here x and y represent the co-ordinates of a particle moving along a streamline ψ fixed with respect to the Earth.) The equations for the coefficients reduced to a form similar to those for the finite-amplitude oscillations of a simple pendulum. Two types of wave-like solutions could be shown to exist. In one of these the streamlines were antisymmetric with regard to the x-axis. In the other the streamlines were distorted sine-waves, which might also be tilted with respect to the east–west direction.

An important study of the effect of various kinds of bottom topography on planetary waves was presented by P. B. Rhines*. The formulation of the problem on a β-plane, with the depth varying in one dimension only, yields a simple Schrödinger-type wave equation. A step-like discontinuity in depth produces a vortex sheet, and even a small discontinuity can cause wave reflexion of order unity. Trapped modes can exist in a region of varying depth. The scattering of plane waves by a sea-mount is of order $(\beta a/\sigma)^3 a$, where a denotes the radius of the island and σ the frequency of the waves. In the case of a two-layer fluid, these same bottom profiles can produce a strong mixing of the barotropic and baroclinic modes. Thus an incident barotropic wave may be transmitted primarily as an upper-layer motion.

Rhines's studies are clearly the starting point for a more realistic study of planetary waves in the oceans. The effect of depth variations on the free oscillation in closed basins might give rise to an interesting and quite tractable series of problems.

A neat experiment to produce Rossby waves in a rotating annulus was described by A. Ibbetson*. Here the β-effect was produced by the varying depth between the horizontal plane bottom and the paraboloidal upper surface of the fluid. The waves were generated by a vertically pivoted wave-maker within the annulus. Two classes of waves appeared: shorter waves propagated towards the 'east' and longer waves towards the 'west', corresponding to their respective group velocities. The wavelengths agreed very well with theoretical predictions, but the damping was somewhat greater than expected. In the course of discussion it was suggested that this might be due to the presence of a surface film. The reflexion of waves from a radial barrier placed in the annulus 'west' of the wave-maker was also studied. At low frequencies the radial motion was confined to a region near the barrier, demonstrating 'westward intensification' of long-period currents.

A contribution to this session by T. V. Davies will be described below. The session ended with two shorter contributions. W. Krauss* discussed the frequency spectra of current velocities in internal waves in the Baltic Sea, at frequencies somewhat greater than the inertial frequency; and G. Chabert d'Hières* described the 150-ton rotating turntable at the University of Grenoble, which has been used for modelling the tides in the English Channel with a remarkable degree of success.

In a later session, A. P. Burger discussed a particular model of a planetary atmosphere, in which it was shown that for oscillations of low frequency the governing differential equation was elliptic in two regions surrounding the poles but hyperbolic in the equatorial zone.

7. Inertial oscillations

It will be recalled that inertial oscillations are non-divergent waves that can exist in the interior of a uniform rotating fluid. In contrast to the planetary waves, all three components of the Coriolis force are brought into play (see the introductory paper by Lighthill*). The waves have a frequency that is always less than or equal to 2Ω. The pressure and the components of velocity satisfy a hyperbolic differential equation, and energy is propagated along the wave crests, which are inclined at an angle $\cos^{-1}(\sigma/2\Omega)$ to the axis of rotation.

In his introductory paper D. Fultz outlined the historical development of the subject, from the 'inertio-elastic' waves of Kelvin to recent experiments on the stability of inertial waves in a rotating basin. He showed a striking film, made in his laboratory, in which the inertial modes were excited in a parabolic rotating basin. As the amplitude of oscillation was increased, certain instabilities became apparent (see figure 5, plate 4).

The inviscid theory of inertial oscillations within a sphere was known to Poincaré (1885); they are described by Legendre polynomials in certain ellipsoidal co-ordinates, with eigenfrequencies given by the roots of a polynomial expression. Recently a viscous theory for the excitation of these oscillations by perturbing

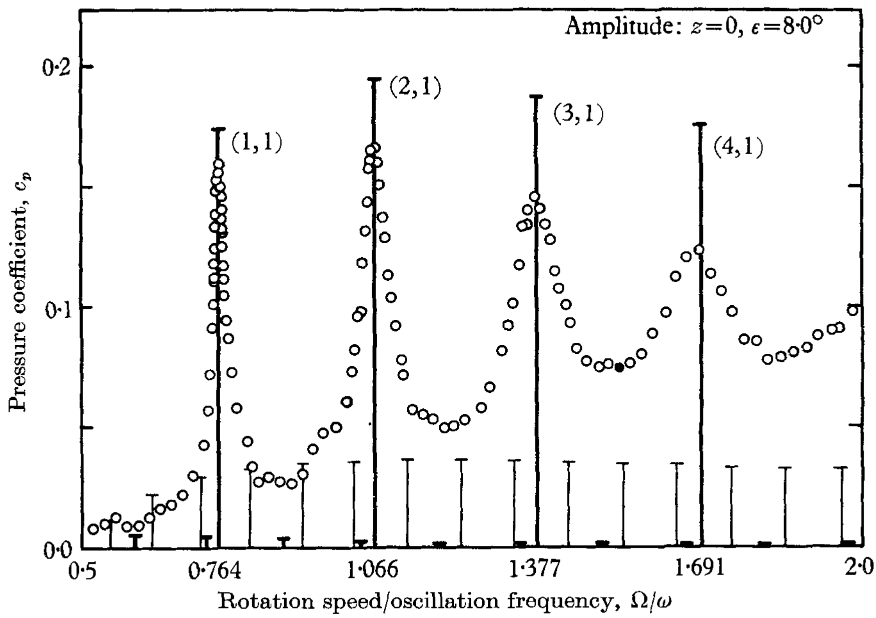

FIGURE 6 (K. Aldridge & A. Toomre). The amplitude of oscillations set up in a spherical container, as detected by the pressure fluctuation at the centre: a comparison of the observed and theoretical frequency response.

the steady rate of rotation of the outer sphere was given by Greenspan (1964). The oscillations are excited by suction in the Ekman layer. It turns out that, though the boundary-layer thickness δ may be very small, the resonant amplitude of the excited oscillation in the interior is of order zero in (δ/R) (R denotes the radius). Some calculations concerning the details of the boundary layers, especially near the critical latitudes ($\sin\theta = \sigma/2\Omega$) were described in a paper presented to the present symposium by G. Venezian.* This was followed by an account of some experiments by K. Aldridge & A. Toomre* in which the theoretical predictions of Greenspan were beautifully verified. A sphere, filled with water, was made to rotate at an almost uniform rate, the angular velocity being perturbed in a sinusoidal fashion. To detect the oscillations, a tube was inserted along the axis of rotation and the pressure measured at the centre (or at some other point on the axis) by the level of water in the tube. The observed frequencies at resonance agreed very well with the theory, and the amplitudes and shape of the response curve were also well predicted—a tribute to the accuracy of the Navier–Stokes equations (see figure 6). The success of the experiment may also be partly due to the fact that the amplitude of the oscillation at resonance is limited by the imposed perturbation of the angular velocity of the sphere. By controlling the latter, non-linear effects can be made uniformly small.

An extraordinary apparatus for insertion in the sphere and said to be a heat exchanger was described by Dr Toomre. It excited some mystification among the audience until it was realized that the meeting was taking place at the beginning of the fourth month of the year.

Considering the success attending experiments in the interior of a single sphere it should be interesting to investigate the oscillations of fluid contained between two concentric spheres. In this case the inviscid theory (as pointed out by Solberg 1936) would predict discontinuous motions along the characteristic cones which touch the inner sphere. A way of investigating solutions in this case (or when the outer surface is free) was suggested in the previous session by T. V. Davies. His method would involve the use of the Green's function for the problem, which is now known. Solutions might be obtained by expanding in Taylor series about the pole of rotation on the one hand and about the equator on the other, and by matching the solutions suitably across the discontinuity. Presumably, however, for small differences between the internal and external radii such solutions would be identical with those found asymptotically by Stern (1963).

Another type of perturbation is produced when a spinning sphere filled with homogeneous fluid is made to precess. W. V. R. Malkus* presented some impressive experimental results, showing that, in the laminar régime, steady zonal flows are set up, and that a zonal jet tends to form at the critical latitudes. He described a non-linear theory, developed in collaboration with F. Busse*, in which the zonal flows are accounted for as second-order rectified flows associated with the (first-order) oscillations of the boundary layer. At large rates of precession the zonal flows themselves exhibit instabilities. These appear earlier if the precession is retrograde relative to the basic rotation. In oblate spheroids, the instabilities induced by precession are more violent, and the flows are more inclined to be turbulent. Complete reversal of flow within the interior can occur. The phenomenon offers a fresh hope for a mechanical explanation of motions in the Earth's core which might be the basis of a self-sustaining dynamo.

The Earth's magnetic field, apart from its dipole component, exhibits certain anomalies (the geomagnetic secular variation) which tend on the whole to drift westward with a speed of the order of 1 mm/day at the Earth's surface. R. Hide* *e*, *f* discussed a tentative theory in which the secular variation would be associated with magnetohydrodynamic oscillations within the liquid core of the Earth. The motions are envisaged as taking place in columns parallel to the Earth's axis. Displacement of a column towards the pole involves stretching the vortex lines as on a β-plane. With rotation alone involved, the phase velocity of such motions would be towards the east, and much too high. If, however, one postulates a toroidal magnetic field within the Earth's core there are two possible modes propagated towards east and west respectively. These are similar in character to Alfven waves. The westwards phase velocity is of the observed magnitude, provided the toroidal field is of the order of 100 G. The eastwards phase velocity is much higher.

Hide's solutions were approximate only, depending on a type of β-plane approximation. In the following paper W. V. R. Malkus* obtained a whole family of solutions in spherical co-ordinates, representing freely propagated modes in a physical system similar to that postulated by Hide. A constant electric current parallel to the axis of rotation (with its associated magnetic field) was taken as the basic state. The normal mode problem then reduced to a modified form of the Poincaré problem. Thus the free modes were described by

Legendre polynomials; but with a different dispersion relation. Reasons were suggested why the westwards travelling modes should be excited in preference to the eastwards travelling modes.

8. A tube model of the ocean

To end the symposium P. Welander* presented a highly simplified, and entertaining, model of the circulation within an ocean basin, in which the flow was assumed to take place only along tubular paths. The system was driven either by 'wind-stresses' or by concentrated heat 'sources' and 'sinks'. Diffusive processes took place only at the joints. In effect the ocean continuum was replaced by a network of tubes (similar to an electrical network) and the differential equations were replaced by a simple set of finite-difference equations. With a system consisting of only three tubes and two joints it was possible to demonstrate, for example, β-effects and the westward intensification of the wind-driven circulation. Even a single loop, with one heat source and one heat sink, exhibited the most surprising sustained or irregular oscillations. The motive behind such a study, namely to reduce a complicated physical system to its simplest terms before studying its behaviour more fully, appeared to be well justified.

Thanks are due to Prof. Munk and to the University of California for their hospitality during the meeting and for the friendly atmosphere and attractive surroundings in which the conference was held. Financial assistance towards the travel expenses of participants was generously provided by the Royal Society and by the U.S. National Science Foundation.

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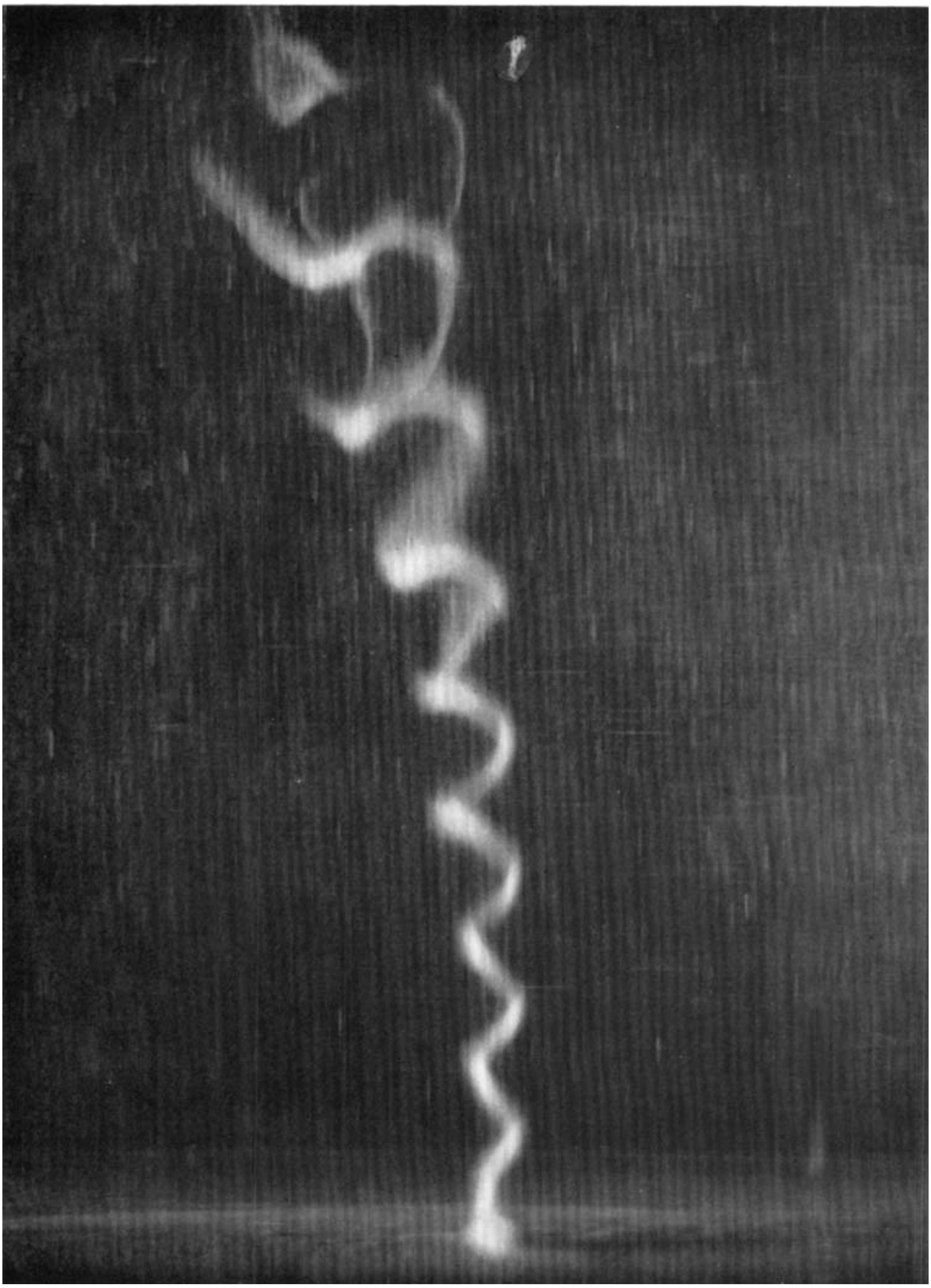

FIGURE 1. (T. Maxworthy). A vertical jet near the axis of rotation, in the flow between two parallel disks. The jet forms at the point of separation of the bottom boundary layer.

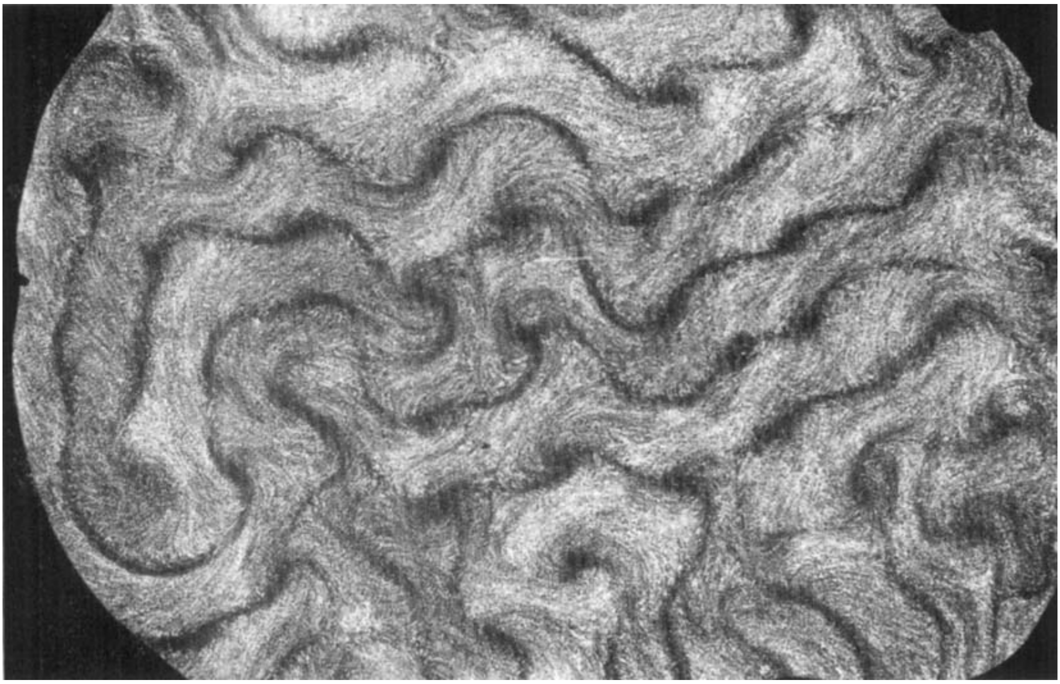

FIGURE 2. (H. T. Rossby). Roll instabilities in a layer of mercury heated from below at a high Taylor number.

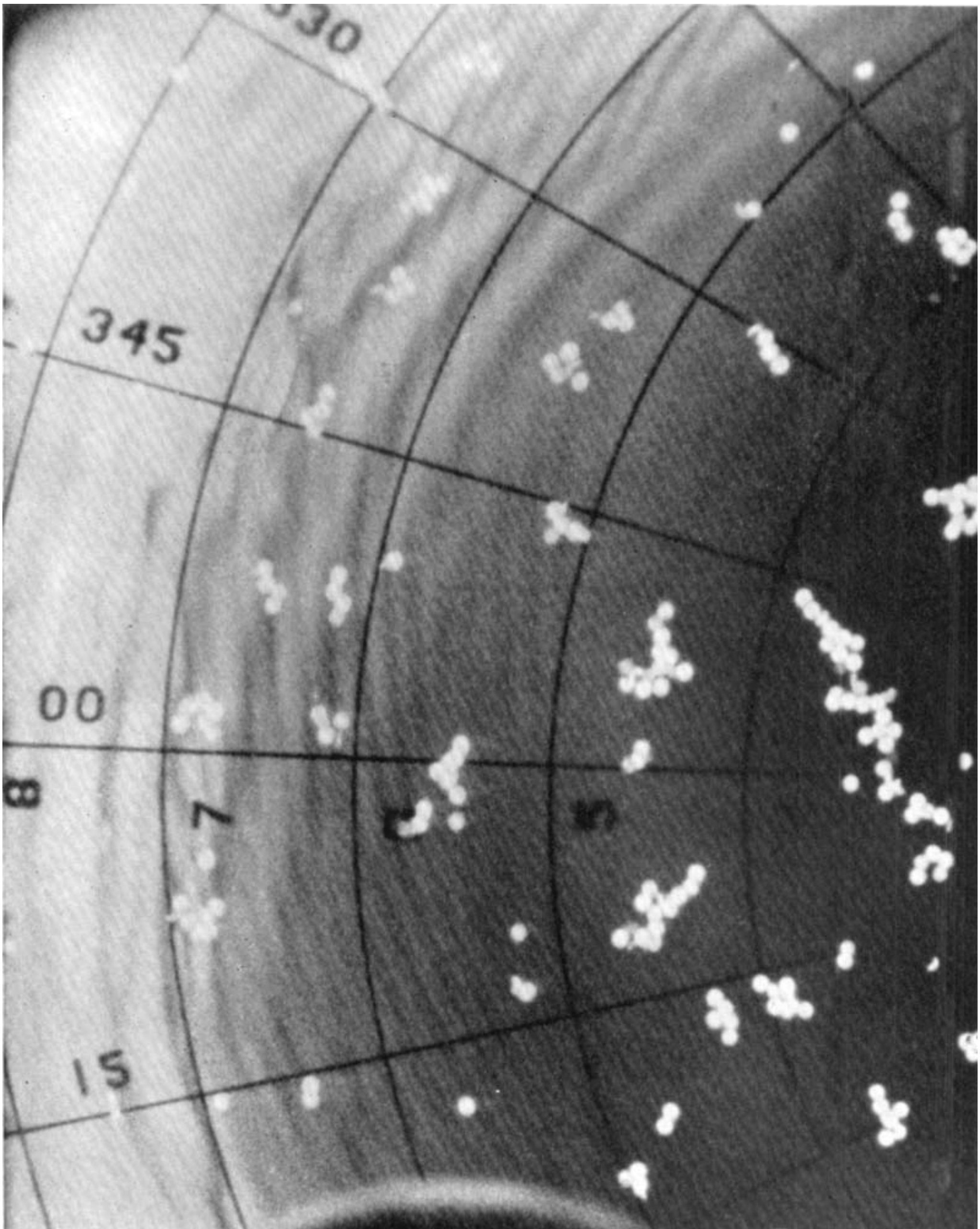

FIGURE 3. (A. J. Faller & R. E. Kaylor). Simultaneous occurrence of types I and II instabilities in the Ekman layer in a rotating cylindrical tank of water. Rotation speed: $0.624\,\text{sec}^{-1}$, accelerated inwards to $1{\cdot}100\,\text{sec}^{-1}$. Boundary-layer depth $= (\nu/\Omega)^{\frac{1}{2}} = 0{\cdot}093\,\text{cm}$. Wavelength of instabilities: Type I, 1·42 cm; type II, 3·54 cm. (The white spots are pieces of paper used as surface tracers.)

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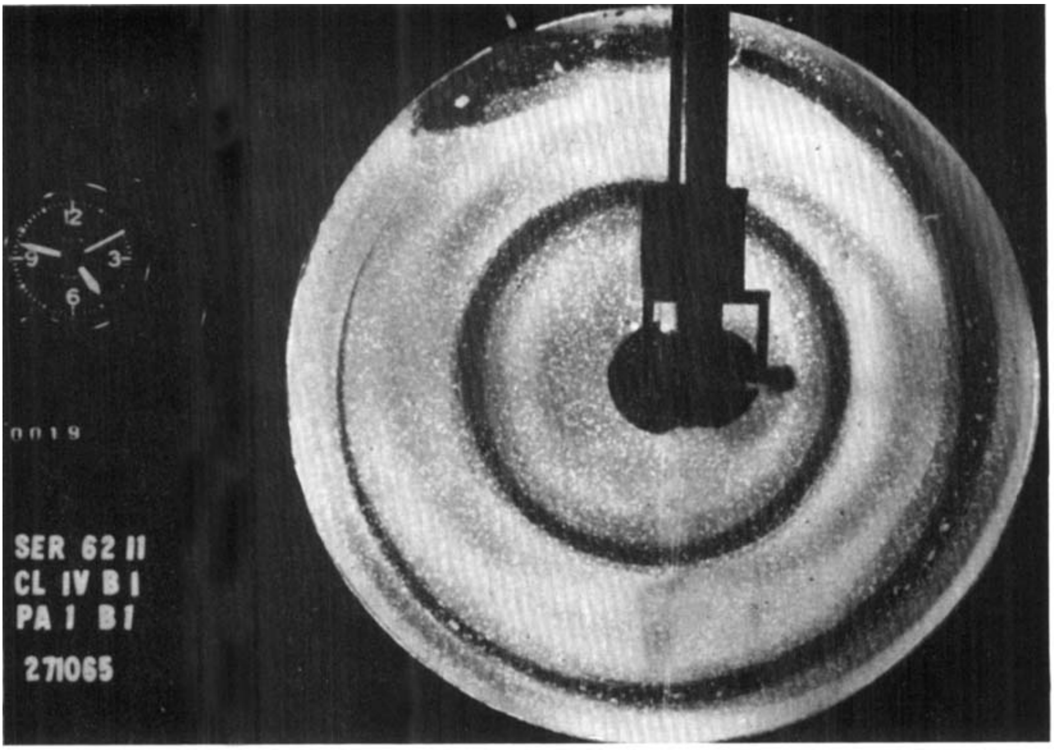
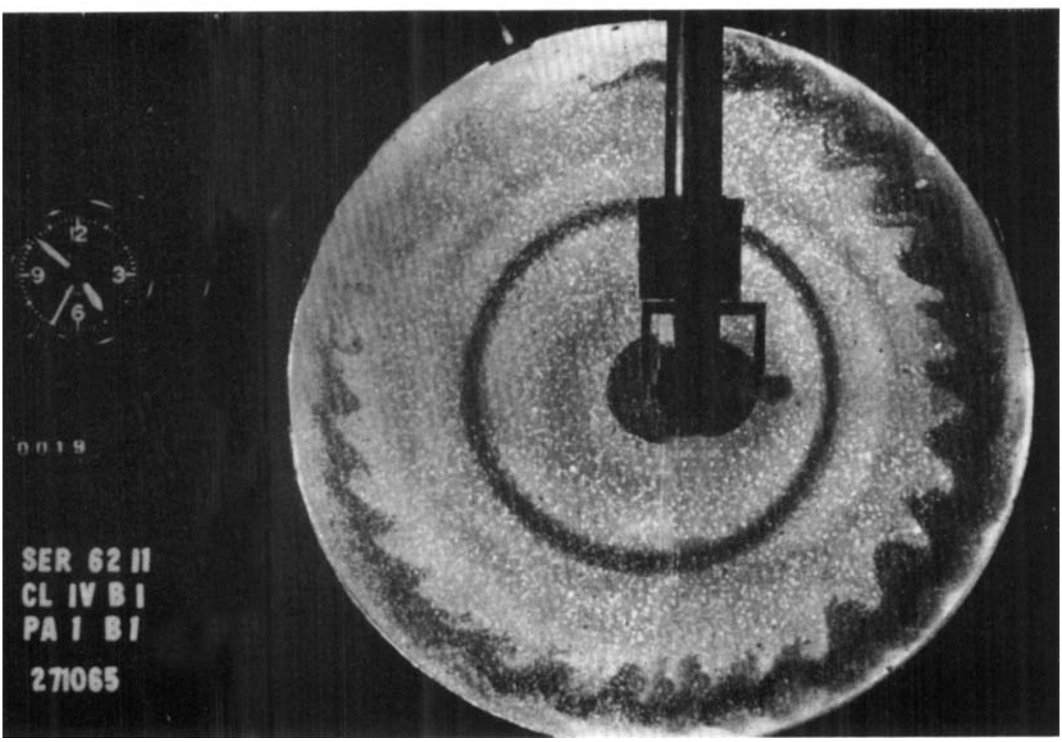


(*a*)



(*b*)

FIGURE 5. (D. Fultz). Inertial oscillations generated in a rotating paraboloidal dish by means of a vertical plunger at the centre (*a*) at a low amplitude of oscillation, (*b*) at a higher amplitude, showing the formation of an instability near the outer edge.

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